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ON GRADED RINGS WITH HOMOGENEOUS DERIVATIONS

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Abstract. We establish results related to homogeneous derivations, a concept introduced by Kanunnikov (2018). First, we prove the existence of a non-trivial family of derivations that are not homogeneous on graded rings. Furthermore, based on homogeneous derivations, we extend certain existing significant results in the context of prime (resp. semiprime) rings to gr-prime (resp. gr-semiprime) rings, such as Posner's and Herstein's theorems.

1. Introduction

This paper is dedicated to studying the commutativity of graded rings via the concept of homogeneous derivations introduced by Kanunnikov in [6]. Over the last 50 years, researchers have been interested in understanding the structure and commutativity of rings using specific types of mappings called derivations. Various authors have widely studied this topic (for example see [1], [5], [10]). The study of commutativity of prime rings with derivations was initiated by Posner in 1957. Since then, the relationship between the commutativity of rings and the existence of specific types of derivations has attracted many researchers. The main result in this context is that a prime ring with a nonzero centralizing derivation must be a commutative ring.

Graded rings have various applications in geometry and physics (for example, see [7]), and appear in various contexts, from elementary to advanced levels. Based on the rich heritage of ring theory, many researchers have attempted to extend and generalize various classical results to graded settings. In this study, we continue to investigate and extend many classical results concerning derivations on prime (resp. semiprime) rings to gr-prime (resp. gr-semiprime) rings.

In this paper, R denotes an associative ring with the center $Z(R)$, and G is an abelian group with identity element e. For $x, y \in R$, we write $[x, y]$ for the Lie product $xy - yx$, and for a nonempty subset S of R, we write $C_R(S) = \{x \in R \mid [x, S] = 0\}$ for the centralizer of S in R. A ring R is G-graded if there is a family $\{R_g, g \in G\}$ of additive subgroups R_g of $(R, +)$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for every $g, h \in G$. The additive subgroup R_g is called the

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homogeneous component of R. The set $\mathcal{H}(R) = \bigcup_{g\in G} R_g$ is the set of homogeneous elements of R. A nonzero element $x \in R_g$ is said to be homogeneous of degree g, and we write $\deg(x) = g$. An element x of R has a unique decomposition $x = \sum_{g\in G} x_g$, with $x_g \in R_g$ for all $g \in G$, where the sum is finite. The x_g terms are called the homogeneous components of the element x. It was proved in [9] that R_e is a subring of R and that if R has a unit 1, then $1 \in R_e$.

Let I be a right (resp. left) ideal of a graded ring R. Then I is said to be a graded right (resp. left) ideal if $I = \bigoplus_{g\in G} I_g$, where $I_g = (I \cap R_g)$ for all $g \in G$. That is, for $x \in I$, $x = \sum_{g\in G} x_g$, where $x_g \in I$ for all $g \in G$. A graded proper ideal P is said to be gr-prime if, whenever $J_1J_2 \subset P$ for J_1 and J_2 graded ideals of R, then $J_1 \subset P$ or $J_2 \subset P$.

A graded ring R is said to be gr-prime (gr-semiprime) if $aRb = \{0\}$ implies $a = 0$ or $b = 0$ (if $aRa = \{0\}$ then $a = 0$), where $a, b \in \mathcal{H}(R)$. Moreover, a graded ring R is a gr-semiprime ring if the intersection of all the gr-prime ideals is zero. This is equivalent to R having no nonzero nilpotent graded ideals. Also a graded ring R is called gr-semiprime if $I^2 \neq \{0\}$ for all nonzero left (right or two-sided) graded ideals I of R. As shown in group rings, there exist gr-prime (gr-semiprime) rings that are not prime (semiprime); for more details we refer to [8]. The reader may find it helpful to keep in mind the implications noted in the following figure:

$$\begin{array}{ccc} \text{prime} & \Longrightarrow & \text{gr-prime} \\ \Downarrow & & \Downarrow \\ \text{semiprime} & \Longrightarrow & \text{gr-semiprime} \end{array}$$

An additive mapping $d : R \longrightarrow R$ is a derivation of a ring R if d satisfies

$$d(xy) = d(x)y + xd(y) \quad \text{(Leibniz formula)}$$

for all $x, y \in R$. A derivation d is called an inner derivation if there exists $a \in R$ such that $d(x) = [a, x]$ for all $x \in R$. A mapping d is called centralizing if

$$[d(x), x] \in Z(R)$$

for all $x \in R$. In particular, if $[d(x), x] = 0$ for all $x \in R$, it is called commuting. Furthermore, it is called central if $d(x) \in Z(R)$ for all $x \in R$.

The remainder of this paper is structured as follows: In Section 2, we review the definition of homogeneous derivations. We give an example of derivation, which is not homogeneous for graded rings. Additionally, we present graded versions of several results from the classical setting which we need for the rest of this paper. In Section 3, we introduce graded analogs of Posner's and Herstein's theorems for graded rings, along with related results. In Section 4, we extend some results given for semiprime rings to gr-semiprime rings.

2. Preliminaries

In this section, we present some results concerning graded rings and homogeneous derivations that will be needed in subsequent sections of this paper.

Proposition 2.1. *Let R be a gr-prime ring. Then the following statements hold:*

(1) *If $aRb = \{0\}$, where a or $b \in \mathcal{H}(R)$, then $a = 0$ or $b = 0$.*
(2) *Let $c \in Z(R) \cap \mathcal{H}(R) \setminus \{0\}$. If x is an element of R such that $cx \in Z(R)$, then $x \in Z(R)$.*
(3) *The centralizer of any nonzero graded one-sided ideal I is equal to the center of R. In particular, if R has a nonzero central graded ideal, then it is a commutative graded ring.*
(4) *Let P be a gr-prime ideal of R. If $aRb \subseteq P$, where a or $b \in \mathcal{H}(R)$, then $a \in P$ or $b \in P$.*

Proof. (1) This follows from the fact that R is a gr-prime ring and that each element of R has a unique decomposition.

(2) Let $c \in Z(R) \cap \mathcal{H}(R) \setminus \{0\}$ and $x \in R$ such that $cx \in Z(R)$. For any $y \in R$, we have $[y, cx] = 0$, which implies $c[y, x] = 0$. Thus, $cR[y, x] = \{0\}$ for all $y \in R$. From (1), we deduce that $[x, y] = 0$ for all $y \in R$. Therefore, $x \in Z(R)$.

(3) Let I be a nonzero graded right ideal of R and $a \in C_R(I)$. By definition, $[a, x] = 0$ for all $x \in I$. This implies $x[a, y] = 0$ for all $x \in I$ and $y \in R$. Hence, $IR[a, y] = \{0\}$. Since I is a nonzero graded ideal, we obtain $rR[a, y] = \{0\}$ for all $y \in R$ and for some $r \in I \cap \mathcal{H}(R) \setminus \{0\}$. Applying (1), we conclude that $[a, y] = 0$ for all $y \in R$. Therefore, $a \in Z(R)$. The reverse implication is immediate.

(4) Let $a, b \in R$. Without loss of generality, assume $b \in \mathcal{H}(R)$ and write $a = \sum_{g\in G} a_g$. Then $aRb \subseteq P$ implies $arb \in P$ for all $r \in \mathcal{H}(R)$, which yields $\sum_{g\in G} a_g rb \in P$. Since P is a graded ideal, we obtain $a_g rb \in P$ for all $g \in G$ and $r \in \mathcal{H}(R)$. Let $I_1 = Ra_g R + Ra_g + a_g R + \mathbb{Z}a_g$ and $I_2 = RbR + Rb + bR + \mathbb{Z}b$. Clearly, I_1 and I_2 are graded ideals of R containing a_g and b respectively. Moreover, $I_1 I_2 \subseteq P$. Since P is a gr-prime ideal, either $I_1 \subseteq P$ or $I_2 \subseteq P$. Thus, $a_g \in P$ or $b \in P$. Therefore, $a \in P$ or $b \in P$. □

Proposition 2.2. *Let R be a gr-semiprime ring. Then the following statements hold:*

(1) *$Z(R)$ contains no nonzero nilpotent homogeneous element.*
(2) *Let $a \in \mathcal{H}(R)$ such that $a[a, x] = 0$ for all $x \in R$, then $a \in Z(R)$.*
(3) *Let I be a graded right ideal of R and*

$$Z(I) = \{x \in I \mid [x, y] = 0 \text{ for all } y \in I\}.$$

Then $Z(I) \subset Z(R)$.

Proof. (1) Suppose that $Z(R)$ contains a nonzero nilpotent homogeneous element r, and let I denote the ideal generated by r. Then I is a nilpotent graded ideal of R. Since R is gr-semiprime, we conclude that $I = \{0\}$, which yields a contradiction.

(2) Let $a \in \mathcal{H}(R)$. For all $x, y \in R$, we have $a[a, xy] = 0$. Then $a\big(x[a, y] + [a, x]y\big) = 0$, which means that $ax[a, y] = 0$ for all $x, y \in R$. This implies that $[a, y]x[a, y] = 0$. So, $[a, y]R[a, y] = \{0\}$ for all $y \in R$. In particular, $[a, r]R[a, r] = \{0\}$ for all $r \in \mathcal{H}(R)$. Since $[a, r] \in \mathcal{H}(R)$ and R is gr-semiprime, it follows that $[a, r] = 0$ for all $r \in \mathcal{H}(R)$. Hence $[a, y] = 0$ for all $y \in R$. Therefore, $a \in Z(R)$.

(3) Let $a \in Z(I) \cap \mathcal{H}(R)$. Then $[a, x] = 0$ for all $x \in I$, which implies $a[a, xy] = 0$ for all $y \in R$ and $x \in I$. Thus, $a[a, y] = 0$ for all $y \in R$. By (2), we conclude that $a \in Z(R)$. Furthermore, for $y = \sum_{g \in G} y_g \in Z(I)$, we have $y_g \in Z(I)$. This implies $y_g \in Z(R)$ for all $g \in G$. Therefore, $y \in Z(R)$. □

Definition 2.3 ([6])**.** Let R be a G-graded ring. An additive mapping $d : R \longrightarrow R$ is called a *homogeneous derivation* if

(i) $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.
(ii) $d(r) \in \mathcal{H}(R)$ for all $r \in \mathcal{H}(R)$.

Remark 2.4. Every homogeneous derivation is a derivation. However, the converse statement is not true, as there exist derivations that are not homogeneous. This can be demonstrated through the following example.

Example 2.5. Let $R = M_2(\mathbb{R})$, where $\mathbb{R}$ is the field of real numbers. Then R is $\mathbb{Z}_4$-graded by

$$R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}, \qquad R_1 = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \mid c \in \mathbb{R} \right\},$$
$$R_3 = \left\{ \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \mid d \in \mathbb{R} \right\}, \qquad R_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

For $x = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$, let $d_x : R \longrightarrow R$ be the inner derivation associated with x. Then d_x is not a homogeneous derivation of R with respect to the $\mathbb{Z}_4$-grading. Indeed, for $r = \begin{pmatrix} 0 & 0 \\ 7 & 0 \end{pmatrix} \in \mathcal{H}(R)$ we have $d_x(r) = \begin{pmatrix} 7 & 0 \\ 7 & -7 \end{pmatrix} \notin \mathcal{H}(R)$.

Lemma 2.6. *Let R be a gr-prime ring and I a nonzero graded left ideal of R. If d is a nonzero homogeneous derivation of R, then its restriction on I is nonzero.*

Proof. Assume that $d(x) = 0$ for all $x \in I$. For any $x \in I$ and $y \in R$, we have $d(y)x = 0$. This implies $d(R)RI = \{0\}$. Since I is a nonzero graded left ideal, we obtain $d(R)Rr = \{0\}$ for some $r \in I \cap \mathcal{H}(R) \setminus \{0\}$. By Proposition 2.1, we conclude that $d(R) = \{0\}$, a contradiction. □

The following proposition is the graded analog of [2, Lemma 4], which we will need to prove Theorem 3.3 and Theorem 4.1.

Proposition 2.7. *Let R be a gr-semiprime ring and I be a nonzero graded left ideal of R. If d is a homogeneous derivation of R which is centralizing on I, then d is commuting on I.*

Proof. The application of arguments analogous to those presented in the proof of [2, Lemma 4] leads to the following:

$$8[d(x), x]^3 = 0$$

for all $x \in I$. In particular, since I is a graded left ideal of R, we have $8[d(r), r]^3 = 0$ for all $r \in I \cap \mathcal{H}(R)$. From Proposition 2.2, it follows that $2[d(r), r] = 0$ for all $r \in I \cap \mathcal{H}(R)$. By straightforward computations, we obtain $[r, d(r)]^3 = 0$ for all $r \in I \cap \mathcal{H}(R)$. Since $[r, d(r)]$ is a central nilpotent homogeneous element of R, Proposition 2.2 implies that $[r, d(r)] = 0$ for all $r \in I \cap \mathcal{H}(R)$. Therefore, $[x, d(x)] = 0$ for all $x \in I$, as required. $\square$

The following lemma is the graded analog of [5, Theorem 1], which we will need to prove Theorem 3.5.

Lemma 2.8. *Let R be a gr-prime ring of characteristic different from 2. If d is a homogeneous derivation and $c \in \mathcal{H}(R)$ is such that*

$$[c, d(x)] \in Z(R)$$

for all $x \in R$, then $d = 0$ or $c \in Z(R)$.

Proof. Firstly, we start with

$$[c, d(x)] = 0 \tag{2.1}$$

for all $x \in R$. Replacing x by xz in (2.1) and using it, we get

$$d(x)[c, z] + [c, x]d(z) = 0 \tag{2.2}$$

for all $x, z \in R$. Substituting $zd(y)$ instead of z in (2.2), it follows that

$$[c, x]zd^2(y) = 0$$

for all $x, y, z \in R$. Hence $[c, x]Rd^2(y) = \{0\}$ for all $x, y \in R$. In particular, $[c, r]Rd^2(y) = \{0\}$ for all $r \in \mathcal{H}(R)$ and $y \in R$. In light of Proposition 2.1, we can see that either $[c, r] = 0$ or $d^2(y) = 0$ for all $r \in \mathcal{H}(R)$ and $y \in R$. This implies that

$$c \in Z(R) \quad \text{or} \quad Rd^2(R) = \{0\}.$$

If $Rd^2(R) = \{0\}$, then $rRd^2(y) = \{0\}$ for all $r \in \mathcal{H}(R) \setminus \{0\}$ and $y \in R$. Using Proposition 2.1, we obtain $d^2(y) = 0$ for all $y \in R$. By computation and noting that $\operatorname{char}(R) \neq 2$, we establish that $d(x)zd(y) = 0$ for all $x, y, z \in R$, which implies $d(x)Rd(y) = \{0\}$ for all $x, y \in R$. Therefore, $d(r)Rd(r) = \{0\}$ for all $r \in \mathcal{H}(R)$. Since R is gr-prime, we get $d(r) = 0$ for all $r \in \mathcal{H}(R)$, hence $d = 0$.

Returning to the general case,

$$[c, d(x)] \in Z(R) \tag{2.3}$$

for all $x \in R$. Writing $[c, x]$ instead of x in (2.3), we obtain

$$\big[c, [d(c), x]\big] \in Z(R) \tag{2.4}$$

for all $x \in R$. Replacing x by cx in (2.4) yields

$$\big[d(c), c\big][c, x] + c\big[c, [d(c), x]\big] \in Z(R) \tag{2.5}$$

for all $x \in R$. By straightforward computations and applying the previously established equations, we obtain $[d(c), c]R\big[[c, x], c\big] = 0$ for all $x \in R$. Proposition 2.1 then implies either $[d(c), c] = 0$ or $\big[[c, x], c\big] = 0$. In the second case, we can see that $[c, x]Rd^2(y) = 0$ holds for all $x, y \in R$. Applying Proposition 2.1 and given that $\operatorname{char}(R) \neq 2$, we conclude that either $c \in Z(R)$ or $d = 0$. Now, assume that $[d(c), c] = 0$. Then (2.5) becomes

$$c\big[c, [d(c), x]\big] \in Z(R)$$

for all $x \in R$. Hence, either $\big[c, [d(c), x]\big] = 0$ or $c \in Z(R)$. In either case, we obtain $\big[c, [d(c), x]\big] = 0$ for all $x \in R$. This can be expressed as $\big[c, \delta_{d(c)}(x)\big] = 0$, where $\delta_{d(c)}$ is a homogeneous derivation defined by $\delta_{d(c)}(z) := [d(c), z]$ for all $z \in R$. From our previous analysis, we conclude that either $c \in Z(R)$ or $\delta_{d(c)} = 0$. Consequently, either $c \in Z(R)$ or $d(c) \in Z(R)$. Both cases force $d(c) \in Z(R)$.

Replacing x by xc in (2.3), we get

$$[c, d(x)]c + [c, x]d(c) \in Z(R) \tag{2.6}$$

for all $x \in R$. Commuting (2.6) with c, we arrive at $\big[[c, x], c\big]d(c) = 0$ for all $x \in R$. Therefore, $\big[[c, x], c\big]Rd(c) = \{0\}$ for all $x \in R$. Applying Proposition 2.1, we obtain that either $\big[[c, x], c\big] = 0$ or $d(c) = 0$. By arguments similar to those previously established, we can see that either $d = 0$ or $c \in Z(R)$. □

3. Some commutativity criteria involving homogeneous derivations on gr-prime rings

We begin this section by introducing the graded analogs of Posner's theorems.

Theorem 3.1. *Let R be a gr-prime ring with characteristic different from 2, and let d_1 and d_2 be derivations of R. Suppose that at least one of d_1 and d_2 is homogeneous and their composition $d_1 d_2$ is a derivation. Then either $d_1 = 0$ or $d_2 = 0$.*

Proof. Without loss of generality, we may assume that d_2 is a homogeneous derivation. By arguments similar to those in the proof of [10, Theorem 1], we obtain the following identity:

$$d_2(x)yd_1(x) + d_1(x)yd_2(x) = 0 \tag{3.1}$$

for all $x, y \in R$. Replacing y by $sd_2(x)t$ in (3.1), we get

$$d_2(x)sd_2(x)td_1(x) + d_1(x)sd_2(x)td_2(x) = 0 \tag{3.2}$$

for all $x, t, s \in R$. Since

$$d_2(x)sd_1(x) = -d_1(x)sd_2(x) \quad \text{and} \quad d_2(x)td_1(x) = -d_1(x)td_2(x),$$

substituting these into (3.2) yields

$$2d_2(x)sd_1(x)td_2(x) = 0$$

for all $x, s, t \in R$. As $\mathrm{char}(R) \neq 2$, we obtain

$$d_2(x)sd_1(x)td_2(x) = 0$$

for all $x, s, t \in R$. Thus, $d_2(x)sd_1(x)Rd_2(x) = \{0\}$ for all $x, s \in R$. In particular, $d_2(r)sd_1(r)Rd_2(r) = \{0\}$ for all $r \in \mathcal{H}(R)$ and $s \in R$. According to Proposition 2.1, we have $d_2(r)sd_1(r) = 0$ or $d_2(r) = 0$ for all $r \in \mathcal{H}(R)$ and $s \in R$. However, $d_2(r) = 0$ also implies $d_2(r)sd_1(r) = 0$. In both cases, $d_2(r)sd_1(r) = 0$ for all $r \in \mathcal{H}(R)$ and $s \in R$, which implies $d_2(r)Rd_1(r) = \{0\}$ for all $r \in \mathcal{H}(R)$. Therefore, either $d_1(r) = 0$ or $d_2(r) = 0$ for all $r \in \mathcal{H}(R)$. Consequently, either $d_1 = 0$ or $d_2 = 0$. □

Here are two examples showing that the two conditions on R given in Theorem 3.1 are necessary and not superfluous.

Example 3.2. (1) Let $R = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$. Then R is $\mathbb{Z}_4$-graded by

$$R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_2 \right\}, \quad R_2 = \left\{ \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \mid b \in \mathbb{Z}_2 \right\}$$

and

$$R_1 = R_3 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Consider the mapping

$$\begin{array}{cccc} d: & R & \longrightarrow & R \\ & \begin{pmatrix} a & b \\ b & a \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}. \end{array}$$

Then d is a homogeneous derivation of R and that R is gr-prime with $\operatorname{char}(R) = 2$. Let us consider $d_1 = d_2 = d$. Then $d_1 d_2$ is a derivation of R. However, neither $d_1 = 0$ nor $d_2 = 0$.

(2) Let $R = M_2(\mathbb{C}) \times M_2(\mathbb{C})$, where $\mathbb{C}$ is the field of complex numbers. Then R is $\mathbb{Z}_4$-graded by

$$\begin{aligned} R_0 &= \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \right) \mid a, b, c, d \in \mathbb{C} \right\}, \\ R_2 &= \left\{ \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \right\}, \\ R_1 &= \left\{ \left(\begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} \right) \mid e, f \in \mathbb{C} \right\}, \\ R_3 &= \left\{ \left(\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \right) \mid g, h \in \mathbb{C} \right\}. \end{aligned}$$

R is not a gr-prime ring. Let us consider two homogeneous derivations

$$\begin{array}{cccc} \delta_1: & R & \longrightarrow & R \\ & (x, y) & \longmapsto & (d_1(x), 0) \end{array} \quad \text{and} \quad \begin{array}{cccc} \delta_2: & R & \longrightarrow & R \\ & (x, y) & \longmapsto & (0, d_2(y)) \end{array}$$

where

$$d_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \quad \text{and} \quad d_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -ib \\ ic & 0 \end{pmatrix}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$. Then $\delta_1 \delta_2$ is a derivation of R. However, $\delta_i \neq 0$ for $i = 1, 2$.

Theorem 3.3. *Let R be a gr-prime ring and I a nonzero graded ideal of R. If R admits a nonzero homogeneous derivation d such that*

$$[d(x), x] \in Z(R)$$

for all $x \in I$, then R is a commutative graded ring.

Proof. We are given that

$$[d(x), x] \in Z(R)$$

for all $x \in I$. According to Proposition 2.7, it follows that $[d(x), x] = 0$ for all $x \in I$. Linearizing this expression yields

$$[d(x), y] = [x, d(y)]$$

for all $x, y \in I$. The identical arguments presented in the proof of [3, Theorem 2.1] result in the following identity:

$$[x, yd(x)] = 0 \tag{3.3}$$

for all $x, y \in I$. Substituting zy for y in (3.3), we get

$$[x, z]yd(x) = 0$$

for all $x, y \in I$ and $z \in R$. This means that $[x, z]RId(x) = \{0\}$ for all $x \in I$ and $z \in R$. Since I is a graded ideal, we have $[r, z]Rr'd(r) = \{0\}$ for all $r, r' \in I \cap \mathcal{H}(R)$ and $z \in R$. In view of Proposition 2.1, we conclude that $[r, z] = 0$ or $r'd(r) = 0$ for all $r, r' \in I \cap \mathcal{H}(R)$ and $z \in R$. Hence $[x, z] = 0$ or $Id(x) = \{0\}$ for all $x \in I$ and $z \in R$. Now, if $[x, z] = 0$ for all $x \in I$ and $z \in R$, then it follows that $I \subset Z(R)$. So, R contains a nonzero central graded ideal. By Proposition 2.1, we conclude that R is a commutative graded ring. If $Id(x) = \{0\}$ for all $x \in I$, then we have $IRd(x) = \{0\}$ for all $x \in I$. In particular, $rRd(x) = \{0\}$ for some $r \in I \cap \mathcal{H}(R) \setminus \{0\}$. Once again invoking Proposition 2.1, it follows that $d(x) = 0$ for all $x \in I$. So, $d(I) = \{0\}$. Moreover, Lemma 2.6 implies that $d = 0$ on R, a contradiction. □

The following example demonstrates that the gr-primeness hypothesis on R is essential.

Example 3.4. Consider $R = \mathbb{C}[X] \times M_2(\mathbb{C})$. R is $\mathbb{Z}$-graded by

$$\begin{aligned}
R_0 &= \mathbb{C} \times \left\{ \begin{pmatrix} z & 0 \\ 0 & z' \end{pmatrix} \mid z, z' \in \mathbb{C} \right\}, \\
R_1 &= \operatorname{span}_{\mathbb{C}}(X) \times \left\{ \begin{pmatrix} 0 & z \\ z' & 0 \end{pmatrix} \mid z, z' \in \mathbb{C} \right\}, \\
R_n &= \begin{cases} \left\{ \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) \right\} & \text{if } n < 0, \\ \operatorname{span}_{\mathbb{C}}(X^n) \times \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} & \text{if } n \geq 2. \end{cases}
\end{aligned}$$

R is not a gr-prime ring. Define a map $d : R \longrightarrow R$ by

$$d\left(P, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(D(P), \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

for all $P \in \mathbb{C}[X]$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$, where $D := \frac{d}{dX}$ is the usual derivation of $\mathbb{C}[X]$. Therefore, d is a homogeneous derivation of R. In addition, the condition $[d(x), x] \in Z(R)$ is satisfied for all $x \in I$ (where $I = R$). However, it should be noted that R is a non-commutative graded ring.

In [6], it was shown that if a gr-prime ring R admits a homogeneous derivation d satisfying

$$[d(x), d(y)] = 0$$

for all $x, y \in R$, then either R is a commutative graded ring or $d^2 = 0$. Moreover, when $\operatorname{char}(R) \neq 2$, it follows that either $d = 0$ or R is a commutative graded ring.

This result is a graded analog of Herstein's theorem [5, Theorem 2]. In light of this result, we examine the more general case where

$$[d_1(x), d_2(y)] \in Z(R)$$

for all $x, y \in R$ and d_1, d_2 are homogeneous derivations.

Theorem 3.5. *Let R be a gr-prime ring of characteristic different from 2. If d_1 and d_2 are nonzero homogeneous derivations of R such that*

$$[d_1(x), d_2(y)] \in Z(R)$$

for all $x, y \in R$, then R is a commutative graded ring.

Proof. In view of our hypothesis, we have

$$[d_1(x), d_2(y)] \in Z(R)$$

for all $x, y \in R$. In particular,

$$[d_1(r), d_2(y)] \in Z(R)$$

for all $r \in \mathcal{H}(R)$ and $y \in R$. According to Lemma 2.8, we have either $d_1(r) \in Z(R)$ for all $r \in \mathcal{H}(R)$ or $d_2(y) = 0$ for all $y \in R$. Hence $d_1(x) \in Z(R)$ for all $x \in R$ or $d_2(R) = \{0\}$. Since $d_2 \neq 0$, it follows that $d_1(x) \in Z(R)$. Moreover, we have $[d_1(x), x] \in Z(R)$ for all $x \in R$. According to Theorem 3.3, we conclude that R is a commutative graded ring. □

The following example shows that both the gr-primeness hypothesis and the condition $\mathrm{char}(R) \neq 2$ imposed on R are necessary.

Example 3.6. (1) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$. R is $\mathbb{Z}_2$-graded by

$$R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \quad \text{and} \quad R_1 = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \mid c \in \mathbb{R} \right\}.$$

Then R is not a gr-prime ring with $\mathrm{char}(R) \neq 2$. Let us consider the following mappings:

$$\begin{array}{cccc} d_1: & R & \longrightarrow & R \\ & \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{cccc} d_2: & R & \longrightarrow & R \\ & \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & a+b-c \\ 0 & 0 \end{pmatrix}. \end{array}$$

Then d_1 and d_2 are homogeneous derivations of R such that $[d_1(x), d_2(y)] \in Z(R)$. However, R is a non-commutative graded ring.

(2) Let $R = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$. Then R is $\mathbb{Z}_4$-graded by

$$R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{Z}_2 \right\}, \quad R_2 = \left\{ \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \mid b \in \mathbb{Z}_2 \right\}, \quad \text{and} \quad R_1 = R_3 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

R is gr-prime with $\mathrm{char}(R) = 2$. Let d be a homogeneous derivation of R defined as follows:

$$\begin{array}{cccc} d: & R & \longrightarrow & R \\ & \begin{pmatrix} a & b \\ b & a \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \end{array}$$

Take $d_1 = d_2 = d$. Then $[d_1(x), d_2(y)] \in Z(R)$ for all $x, y \in R$. However, R is a non-commutative graded ring.

If R is equipped with a trivial grading, we obtain a generalization of Herstein's result [5]. As an immediate consequence of Theorem 3.5, we have the following corollary.

Corollary 3.7. *Let R be a prime ring of characteristic different from 2. If R admits two nonzero derivations d_1 and d_2 such that*

$$[d_1(x), d_2(y)] \in Z(R)$$

for all $x, y \in R$, then R is a commutative ring.

In [1], it was proved that a prime ring R with a nonzero ideal I must be commutative if it admits a derivation d satisfying either of the following properties:

$$d(x)d(y) \pm xy \in Z(R)$$

for all $x, y \in I$. Inspired by this result, we aim to investigate a more general setting by considering differential identities involving two homogeneous derivations d_1 and d_2 satisfying

$$d_1(x)d_2(y) \pm xy \in Z(R)$$

for all $x, y \in I$, where I is a graded ideal of a gr-prime ring R.

Theorem 3.8. *Let R be a gr-prime ring and I a nonzero graded ideal of R. If R admits two homogeneous derivations d_1 and d_2 satisfying*

$$d_1(x)d_2(y) \pm xy \in Z(R)$$

for all $x, y \in I$, then R is a commutative graded ring.

Proof. Consider the case

$$d_1(x)d_2(y) - xy \in Z(R) \tag{3.4}$$

for all $x, y \in I$. Next, we examine two cases.

Case 1: If $d_1 = 0$ or $d_2 = 0$, we have $xy \in Z(R)$ for all $x, y \in I$. This implies that $[xy, z] = 0$ for all $x, y \in I$ and $z \in R$. In particular,

$$[xy, x] = 0 \tag{3.5}$$

for all $x, y \in I$. Substituting yz for y in (3.5), we obtain

$$xy[z, x] = 0$$

for all $x, y, z \in I$. Therefore, $xRI[z, x] = \{0\}$ for $x, z \in I$. Since I is a nonzero graded ideal of R, we have $rRI[z, r] = \{0\}$ for all $r \in I \cap \mathcal{H}(R) \setminus \{0\}$ and $z \in I$. In light of Proposition 2.1, we obtain $I[z, r] = \{0\}$ for all $r \in I \cap \mathcal{H}(R)$ and $z \in I$, which implies that $I[x, z] = \{0\}$ for all $x, z \in I$. Thus, $IR[z, x] = \{0\}$ for all $x, z \in I$. Applying Proposition 2.1, we conclude that I is commutative. Hence, I is a central graded ideal of R, and consequently, R is a commutative graded ring.

Case 2: If $d_1 \neq 0$ and $d_2 \neq 0$, replacing x by xz in (3.4), we get

$$d_1(x)zd_2(y) + x\big(d_1(z)d_2(y) - zy\big) \in Z(R) \tag{3.6}$$

for all $x, y, z \in I$. Commuting (3.6) with x, we obtain

$$[d_1(x)zd_2(y) + x\big(d_1(z)d_2(y) - zy\big), x] = 0$$

for all $x, y, z \in I$, which means that

$$[d_1(x)zd_2(y), x] = 0 \tag{3.7}$$

for all $x, y, z \in I$. Substituting zt for z in (3.7), we get

$$d_1(x)zt[d_2(y), x] + [d_1(x)zt, x]d_2(y) = 0 \tag{3.8}$$

for all $x, y, z \in I$ and $t \in R$. Substituting $d_2(y)$ for t in (3.8) and using (3.7), we obtain

$$d_1(x)zd_2(y)[d_2(y), x] = 0$$

for all $x, y, z \in I$. Thus,

$$d_1(x)RId_2(y)[d_2(y), x] = \{0\}$$

for all $x, y \in I$. In particular,

$$d_1(r)RId_2(y)[d_2(y), r] = \{0\}$$

for all $y \in I$ and $r \in I \cap \mathcal{H}(R)$. According to Proposition 2.1, we have either $d_1(r) = 0$ or $Id_2(y)[d_2(y), r] = \{0\}$. Therefore, $d_1(x) = 0$ or $Id_2(y)[d_2(y), x] = \{0\}$ for all $x, y \in I$. Since $d_1 \neq 0$, we have $d_1 \neq 0$ on I. Hence $Id_2(y)[d_2(y), x] = \{0\}$ for all $x, y \in I$, which implies that

$$IRd_2(y)[d_2(y), x] = \{0\}$$

for all $x, y \in I$. So,

$$d_2(y)t[x, d_2(y)] = 0$$

for all $x, y, t \in I$. This implies that

$$d_2(y)RI[x, d_2(y)] = \{0\}$$

for all $x, y \in I$. Using arguments similar to those above, we can show that $[s, d_2(y)] = 0$ for all $y \in I$ and $s \in R$. Hence $d_2(I) \subset Z(R)$. Moreover, we have $[xy, d_1(x)] = 0$ for all $x, y \in I$. Substituting yt for y, where $t \in R$, we get $xy[t, d_1(x)] = 0$. So, $xRI[t, d_1(x)] = \{0\}$. It follows from Proposition 2.1 that $[t, d_1(x)] = 0$ for all $x \in I$ and $t \in R$. Hence $d_1(I) \subset Z(R)$. Therefore, (3.4) reduces to

$$xy \in Z(R)$$

for all $x, y \in I$. According to Case 1, we conclude that R is a commutative graded ring. Further, in the end the second case $d_1(x)d_2(y) + xy \in Z(R)$ can be reduced to the first one by considering $-d_2$ instead of d_2. $\square$

The following example shows that the gr-primeness condition imposed on R is necessary.

Example 3.9. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \times \mathbb{R}[X]$. Then R is $\mathbb{Z}$-graded by

$$R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\} \times \mathbb{R}, \quad R_1 = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{R} \right\} \times \mathrm{span}_{\mathbb{R}}(X)$$

and

$$R_n = \begin{cases} \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \times \mathrm{span}_{\mathbb{R}}(X^n) & \text{if } n \geq 2, \\ \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \times \left\{ 0_{\mathbb{R}[X]} \right\} & \text{if } n < 0. \end{cases}$$

Let us consider the following mappings:

$$\begin{array}{rrcl} d_1 : & R & \longrightarrow & R \\ & \left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, P \right) & \longmapsto & \left(\begin{pmatrix} 0 & 5b \\ 0 & 0 \end{pmatrix}, \frac{dP}{dX} \right) \end{array}$$

and

$$\begin{array}{rrcl} d_2 : & R & \longrightarrow & R \\ & \left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, P \right) & \longmapsto & \left(\begin{pmatrix} 0 & a+2b \\ 0 & 0 \end{pmatrix}, 0_{\mathbb{R}[X]} \right) \end{array}$$

Then d_1 and d_2 are homogeneous derivations of R. Consider the graded ideal $I = \left\{ \begin{pmatrix} 0 & 3c \\ 0 & 0 \end{pmatrix} \mid c \in \mathbb{R} \right\} \times \langle X^2 \rangle$ of R. Here, R is not a gr-prime ring and $d_1(x)d_2(y) \pm xy \in Z(R)$ for all $x, y \in I$. However, R is a non-commutative graded ring.

4. On graded ideals of gr-semiprime rings involving homogeneous derivations

In [2], it was shown that if a semiprime ring R admits a derivation that is centralizing on some nontrivial one-sided ideal, then R must have a nontrivial central ideal. Moreover, under similar conditions, prime rings are proven to be commutative. The following theorem extends these results to gr-semiprime rings.

Theorem 4.1. *Let R be a gr-semiprime ring and I a nonzero graded left ideal of R. If R admits a homogeneous derivation d which is nonzero on I and centralizing on I, then R contains a nonzero central graded ideal.*

Proof. By Proposition 2.7, d is commuting on I, i.e., $[d(x), x] = 0$ for all $x \in I$. A linearization of this relation leads to

$$[y, d(x)] + [x, d(y)] = 0 \tag{4.1}$$

for all $x, y \in I$. Writing yx instead of y in (4.1) and using it, we get

$$[x, y]d(x) = 0 \tag{4.2}$$

for all $x, y \in I$. Replacing y by zy in (4.2), we get

$$[x, z]yd(x) = 0$$

for all $x, y, z \in I$, which implies that

$$[x, z]RId(x) = \{0\}$$

for all $x, z \in I$. Since R is gr-semiprime, there exists a family $\mathcal{F} := \{P_i : i \in \Lambda\}$ of gr-prime ideals such that $\bigcap_{i\in\Lambda} P_i = \{0\}$, and therefore,

$$[x, z]RId(x) \subset P_i$$

for all $x, z \in I$ and $i \in \Lambda$. Since I is a graded left ideal, it follows that for each $i \in \Lambda$, we have

$$[r', z]Rrd(r') \subseteq P_i$$

for all $r, r' \in I \cap \mathcal{H}(R)$ and $z \in I$. By Proposition 2.1, we have either $[r', z] \in P_i$ or $rd(r') \subset P_i$ for all $r, r' \in I \cap \mathcal{H}(R)$ and $z \in I$. Therefore, we claim that

$$[x, z] \in P_i \quad (*)$$

for all $x, z \in I$, or

$$Id(I) \subset P_i \quad (**)$$

for all $i \in \Lambda$. Let $\Gamma_1 := \{i \in \Lambda \mid [x, z] \in P_i \text{ for all } x, z \in I\}$ and $\Gamma_2 := \{i \in \Lambda \mid Id(I) \subset P_i\}$. Call P_i a type $(*)$ gr-prime for all $i \in \Gamma_1$, and a type $(**)$ gr-prime for all $i \in \Gamma_2$. Let P_{Γ_1} and P_{Γ_2} be the intersections of all type $(*)$ and type $(**)$ gr-primes, respectively. Note that $P_{\Gamma_1} \cap P_{\Gamma_2} = \{0\}$. Arguments identical to those used in the proof of [2, Theorem 3] and the fact that P_{Γ_2} is gr-prime lead to

$$Rd(R) \subset P_{\Gamma_2}.$$

Let us consider J, the left ideal generated by $d(R)I$. Then J is a graded left ideal. We need to show that J is a nonzero graded two-sided central ideal. First, let us prove that J is nonzero. If $J = \{0\}$, then $d(R)I = \{0\}$. Since $Id(R)$ is a graded left ideal of R and $Id(R)Id(R) = \{0\}$, we have $(Id(R))^2 = \{0\}$. As a gr-semiprime ring contains no nonzero nilpotent graded left ideal, we obtain $Id(R) = \{0\}$. Thus, $yd(zz') = 0$ for all $y \in I$ and $z, z' \in R$, which implies $yd(z)z' + yzd(z') = 0$ and consequently $yzd(z') = 0$. Therefore,

$$IRd(R) = \{0\}. \tag{4.3}$$

Straightforward computations with (4.3) show that $d(y)Rd(y) = \{0\}$ for all $y \in I$. In particular, $d(r)Rd(r) = \{0\}$ for all $r \in I \cap \mathcal{H}(R)$. By the gr-semiprimeness of R, we obtain $d(r) = 0$ and hence $d(I) = \{0\}$, which contradicts our assumption. Therefore, we conclude that J must be nonzero. Using the same arguments as in the proof of [2, Theorem 3] and the above results, we can show that J is commutative. Thus, $J = Z(J)$. According to Proposition 2.2, we have $J \subseteq Z(R)$, which implies that J is a central graded ideal. □

As a direct consequence of Proposition 2.2 and Theorem 4.1, we have the following corollary.

Corollary 4.2. *Let R be a gr-prime ring and I a nonzero graded left ideal. If R admits a nonzero homogeneous derivation that is centralizing on I, then R is a commutative graded ring.*

Theorem 4.3. *Let R be a gr-semiprime and d a homogeneous derivation of R such that $d^3 \neq 0$ and $[d(x), d(y)] = 0$ for all $x, y \in R$. Then R contains a nonzero central graded ideal.*

Proof. Let R^d denote the subring of R generated by elements of the form $d(x)$, where $x \in R$. We have $d(R) \subset R^d$. Since $d^3 \neq 0$, there exists a homogeneous element $r \in \mathcal{H}(R) \setminus \{0\}$ such that $d^3(r) \neq 0$. So, $d^2(d(r)) \neq 0$. Let $y = d(r) \in R^d$ and let I be the ideal generated by $d^2(y)$. Clearly, I is contained in R^d. Since $d^2(y) \in \mathcal{H}(R)$, it follows that I is a graded ideal. Consequently, R^d contains a nonzero graded ideal I of R. Based on our hypothesis, I is commutative, and by Proposition 2.7, we have $I = Z(I) \subset Z(R)$. Hence I is a nonzero central graded ideal of R. □

In [4], Daif extends a result of Herstein concerning a derivation d on a prime ring R satisfying

$$[d(x), d(y)] = 0$$

for all $x, y \in R$, to the case of semiprime rings. In what follows, we extend this result to gr-semiprime rings.

Theorem 4.4. *Let R be a gr-semiprime ring and I a nonzero graded ideal of R. Assume that R admits a homogeneous derivation d which is nonzero on I and satisfies*

$$[d(x), d(y)] = 0$$

for all $x, y \in I$. Then R contains a nonzero central graded ideal.

Proof. By hypothesis, we have

$$[d(x), d(y)] = 0 \tag{4.4}$$

for all $x, y \in I$. Replacing y by yz in (4.4), we obtain

$$d(y)[d(x), z] + [d(x), y]d(z) = 0 \tag{4.5}$$

for all $x, y, z \in I$. Substituting zs for z in (4.5), we get

$$d(y)z[d(x), s] + [d(x), y]zd(s) = 0$$

for all $x, y, z \in I$ and $s \in R$. Replacing s with $d(t)$, we find that

$$[d(x), y]zd^2(t) = 0$$

for all $x, y, z, t \in I$. Now, using the same arguments as in the proof of Theorem 4.1, we arrive at $[d(x), y] = 0$ for all $x, y \in I$. In particular, $[d(x), x] = 0$ for all $x \in I$. By Theorem 4.1, R contains a nonzero central graded ideal. □

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